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THE STRUCTURE OF STRESS–STRAIN RELATIONS IN FINITE ELASTO–PLASTICITY

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Abstract—The importance of the distinction between rotation in the continuum and material sense is emphasized. Starting from the stability of thin-walled tube test specimens, material stability at finite strains is formulated for any choice of conjugate stress and strain. Normality is investigated when total strain rates are non-integrable. It is the total strain rate minus the instantaneous elastic strain rate that follows normality. The general lack of normality exhibited by the rate of strain of the unloaded body element is evaluated quantitatively. A general criterion for convexity of the yield surface is established, and the effect on convexity of changing the stress definition is examined.

1. INTRODUCTION

Elastic–plastic behavior over large changes of plastic strain is of engineering importance in a variety of applications and is of considerable physical and mathematical interest as well. Problems such as necking or strain localization can be very sensitive to the form of the adopted constitutive law[1]. Physical understanding of elastic–plastic deformation is necessary, delineating the mathematical framework of permissible stress–strain relations is of prime importance.

Earlier work by Hill and Rice[2, 3] employed symmetric, conjugate measures of stress and strain. The plastic strain rate was defined appropriately as the total strain rate minus the instantaneous elastic strain rate. Normality then was established from Ilyushin's postulate[4] The case when the strain rate conjugate to stress cannot be integrated into a strain measure has not previously been investigated . Convexity of the yield surface in stress space or lack thereof has been little explored. Results have been obtained for special cases only[5, 6]. The definition of plastic strain or plastic strain rate itself still is a controversial topic[7, 8]. Physical arguments have been given to consider the permanent or residual strain of the material element in its unloaded state as plastic[9–11], rather than to subtract the elastic component of the strain rate from the total to obtain the plastic.

The purpose of the present paper is to shed more light on the structure of stress–strain relations in isothermal, finite elasto–plasticity. Elementary microstructural and atomic scale considerations bring out the difficulty that arises because the simplest reference state is very different for elastic and plastic responses. A discussion of the simple tension or compression test illustrates the intrinsic suitability of nominal or Lagrangian stress for purely elastic response and of true or Eulerian stress for purely plastic response. The need to make a clear distinction between rotation in the continuum and material sense is demonstrated by the consideration of torsion of a thin-walled tube. Although repeated reference is made to the physical nature of elastic and plastic deformation, most of what follows is primarily mathematical in nature. The starting point is to assume stability (in the special sense of Drucker)[12–16] for thin-walled tubes under combined tension, torsion and interior pressure over all load cycles beginning inside the elastic domain and involving infinitesimal plastic straining. The generalization to arbitrary states of stress gives material stability in terms of any choice of symmetric, conjugate stress and strain measures. The dependence on the chosen stress definition of the additive decomposition of strain rate into elastic and plastic components is discussed quantitatively. Using this decomposition, and building on[2, 3], lack of normality is exhibited and evaluated when strain rates are non-integrable. Mathematical flexibility is provided by attaching a single parameter to each plastic loading. A general, sufficient condition for convexity of the yield surface is established. Otherwise possible deviations from convexity are given a bound. The effect on convexity of changing the stress definition is calculated independently of stability considerations. The possible use of nonsymmetric stresses of the Piola type is discussed. It is pointed out that the rate of the permanent or residual strain of the body element in its stress-free state does not follow normality in general. The difference between the permanent or residual strain rate and the plastic strain rate as defined here is evaluated explicitly.

2. SIMPLE TENSION AND COMPRESSION AND LAGRANGIAN VERSUS EULERIAN CHOICE

The difficulty of simultaneously modelling the elastic and plastic modes of deformation is illustrated by the extension or compression of a bar under an applied axial force F.

In a purely elastic response the atoms in the cross-section of the bar get closer if the bar is stretched and wider apart if the bar is compressed axially. Their number does not change provided the atomic configuration remains stable. The unstressed initial cross-section A_0 is representative of this number of atomic chains carrying the axial load. Therefore, the nominal or Lagrangian stress (F/A_0) is well suited for constitutive equations at large elastic strain.

On the contrary, large plastic deformation greatly decreases the number of atoms and therefore of atomic bonds carrying the load in tension and greatly increases the number in compression. The current cross-sectional area A now is much more appropriate than A_0 although the area upon elastic unloading of the bar would seem better. When the elastic deformation is small compared with the plastic, the Cauchy or Eulerian stress (F/A) provides a good measure of the driving force.

3. THIN-WALLED TUBE TESTS

A proper definition of rotation is essential for stress–strain relations when shearing is present. The usual continuum approach is appropriate for purely elastic deformation. Rotation is defined from the polar decomposition of the deformation gradient tensor[17]. However, for plastic deformation of crystalline solids the often quite different rotation of the lattice is what matters for the material[11].

The distinction between the rotation in the continuum and in the material sense is brought into sharp focus by the torsion of macroscopically homogeneous thin-walled tubes, Fig. 1. The lattice or material rotation about the radial direction nearly vanishes because plane cross-sections of the tube must remain macroscopically plane and so the planes on which slip occurs cannot rotate on average. The continuum approach however gives a spin of half the shear strain rate. The physical impropriety of including this spin or the overall continuum rotation in an assumed isotropic stress–strain relation is clear because the principal axes of stress do not rotate and the overall geometry of the body remains unaltered. Any instabilities calculated to occur because of such rotation are but artifacts of the assumption.

Only statically determinate test specimens can provide directly interpretable experimental information on material behavior. Knowledge of the geometry and the loads gives the stresses unambiguously, however they are defined, and the strains can be determined by measuring the geometry change. Thin-walled tubes subjected to twisting moment M_T, axial force F, and interior pressure P, Fig. 1, are best, provided the material is homogeneous on the macroscale and permits the construction of tubes with axisymmetric properties. Geometry is entirely specified by the angle of twist Φ of the gage section, length L, diameter D, and wall thickness t.

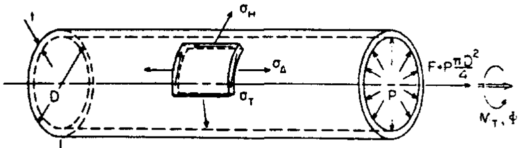

Fig. 1. Thin-walled tube test specimen.

Yet information is limited to plane stress and restricted to a moderate strain range from any given state of plastic deformation over which the response is stable. Care must be taken not to take the traditional experimental results for nominal conventional stresses and strains to apply instead to a very different choice of stress and conjugate strain. The stress–strain relation that would result would not reproduce the response of the test from which it was derived and might incorrectly predict instability in the stable range. The starting point here is to assume the thin-walled tube to be stable[12–16] for all load paths (including cycles) beginning inside the elastic domain and involving infinitesimal plastic straining. Let M_T^o, F^o, P^o be the forces at the beginning of a particular load path. With working displacements $\mathrm{d}\Phi$ for M_T, $\mathrm{d}L$ for F, and $\mathrm{d}V$, the change of the interior volume of the gage section, for P, the work done by the added forces over the load path or cycle must be positive,

$$\int [(M_T - M_T^o)\,\mathrm{d}\Phi + (F - F^o)\,\mathrm{d}L + (P - P^o)\,\mathrm{d}V] \geq 0. \tag{3.1}$$

Equation (3.1) gives normality of the plastic increments $\mathrm{d}\Phi^p$, $\mathrm{d}L^p$, $\mathrm{d}V^p$ that remain after a cycle of addition and removal of $\mathrm{d}M_T$, $\mathrm{d}F$, $\mathrm{d}P$ at yield. Convexity of the yield surface in load space also follows for negligible changes in the elastic response of the tube[14].

4. MATERIAL STABILITY

The next step is to convert (3.1), applied to cycles of load, into a material stability form. Symmetric, conjugate measures of stress σ_{ij} and strain ϵ_{ij}[2] provide the appropriate framework. Conjugate means that $\sigma_{ij}\dot{\epsilon}_{ij}$ is the rate of work per unit reference volume. For the sake of simplicity, but with no loss of generality, rectangular Cartesian coordinates are used throughout the text. The usual and reasonable assumption is made that a piecewise regular yield surface of finite size exists in load or stress space. Unstable behavior is allowed in the form of a gradually falling stress–strain curve in a uniaxial test. Discontinuous drops in stress are ruled out in this discussion. More generally, a smooth inward motion of the yield surface in stress space is permitted. Let σ_T, σ_A, σ_H be any stress measure in the tube specimen, Fig. 1, with conjugate strains ϵ_T, ϵ_A, ϵ_H. From arguments developed below (3.1) is equivalent to

$$\int [(\sigma_T - \sigma_T^o)\,\mathrm{d}\epsilon_T + (\sigma_A - \sigma_A^o)\,\mathrm{d}\epsilon_A + (\sigma_H - \sigma_H^o)\,\mathrm{d}\epsilon_H] \geq 0 \tag{4.1}$$

over the corresponding stress cycle. Extension of (4.1) to the unstable range of response of the thin-walled tube and to general states of stress characterizes the class of materials to be considered here. Choosing arbitrary stress measure, conjugate strain, and fixed reference state, the materials under present investigation are such that the work done by the added stresses

$$\int (\sigma_{ij} - \sigma_{ij}^o)\,\mathrm{d}\epsilon_{ij} \geq 0 \tag{4.2}$$

over any stress cycle starting at σ_{ij}^o inside the elastic domain and involving infinitesimal plastic straining, Fig. 2.

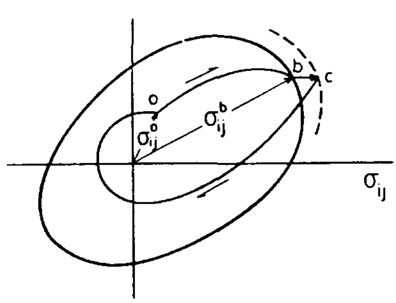

Fig. 2. Stress cycle with infinitesimal plastic loading.

Ilyushin's postulate[4] over cycles of displacement or strain has been used under finite strain conditions instead of structural or material stability. In one sense there is little significant difference between the two approaches for an elastic–plastic body or material because the work done over a cycle of strain starting inside the yield surface and involving infinitesimal plastic straining is the work done by the added stresses over the same path. It differs from the work of the added stresses over a closed cycle of stress by second order terms in the plastic strain increment[16]. Yet as the elastic response tends to zero, the plastic response dominates and the development in the displacement or strain space becomes less satisfactory. Cycles of displacement or strain require large changes in load or stress. In the rigid plastic limit such cycles are no longer possible without reverse plastic deformation. When the earlier picture of Drucker is employed of paths in load or stress space and of stability in such spaces the vanishing of elastic response makes the nature of the plastic response much clearer. Equation (4.2) becomes simply

$$(\sigma_{ij}^b - \sigma_{ij}^o)\dot{\epsilon}_{ij}^p \geq 0. \tag{4.3}$$

Normality of $\dot{\epsilon}_{ij}^p$ and convexity of the yield surface in stress space follow directly. Elastic response and changes in strain energy accompany but cannot be the significant feature of plasticity theory.

A strain energy function is employed to describe the elastic response within the yield surface. It can be viewed as a functional of the whole history of loading[2]. Yet the approach here is to restrict attention to a single, small loading into the plastic range such as bc in Fig. 2. The functional dependence is replaced by a monotonically increasing parameter p attached to the successive states of the material along bc. The formalism in [2] is accordingly modified as follows. Elastically unloading the material at each p defines a sequence of strain energy functions $\phi(\epsilon_{ij}, p)$, taken to be twice continuously differentiable, such that

$$\sigma_{ij} = \frac{\partial \phi}{\partial \epsilon_{ij}}(\epsilon_{kl}, p) = \sigma_{ij}(\epsilon_{kl}, p). \tag{4.4}$$

The tensor of elastic moduli $(\partial\sigma_{ij}/\partial\epsilon_{kl})$ is assumed non-singular to allow inversion of the stress–strain relations (4.4). The complementary potential is

$$\psi(\sigma_{ij}, p) = \sigma_{kl}\epsilon_{kl} - \phi(\epsilon_{ij}, p) \tag{4.5}$$

from which

$$\epsilon_{ij} = \frac{\partial \psi}{\partial \sigma_{ij}}(\sigma_{kl}, p) = \epsilon_{ij}(\sigma_{kl}, p). \tag{4.6}$$

For normality to hold the plastic strain rate is defined as the total strain rate minus the elastic strain rate computed from $\dot{\sigma}_{ij}$ and the instantaneous elastic compliances at the current state of stress. Equation (4.6) applies not only to the conceptual elastic unloadings but also to bc itself for which

$$\dot{\epsilon}_{ij} = \frac{\partial^2 \psi}{\partial \sigma_{ij}\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial}{\partial p}\left(\frac{\partial \psi}{\partial \sigma_{ij}}\right)\dot{p}. \tag{4.7}$$

The first term on the r.h.s. of (4.7) is the elastic strain rate as defined above. Hence the second term is the plastic strain rate

$$\dot{\epsilon}_{ij}^p = \frac{\partial}{\partial p}\left(\frac{\partial \psi}{\partial \sigma_{ij}}\right)\dot{p} = \frac{\partial}{\partial \sigma_{ij}}\left(\frac{\partial \psi}{\partial p}\right)\dot{p}. \tag{4.8}$$

Physically, the plastic strain increment $d\epsilon_{ij}^p$ is what remains after addition and removal of an increment of stress $d\sigma_{ij}$ at yield provided the yield surface in stress space moves locally outwards to allow the elastic closure of this infinitesimal stress cycle, Fig. 2. What is taken to be plastic depends upon the chosen stress definition as cycling one stress measure does not necessarily cycle another. This effect is evaluated as follows. Let σ_{ij}, ϵ_{ij} and σ^*_{ij}, ϵ^*_{ij} be two systems of symmetric stresses and strains. ϵ_{ij} and ϵ^*_{ij} are purely geometric and so are related by a one-one correspondence. Let ρ_0 and ρ^*_0 be the mass densities in the reference states of both systems. The rate of work per unit mass is

$$\frac{1}{\rho_0}\sigma_{kl}\dot{\epsilon}_{kl} = \frac{1}{\rho^*_0}\sigma^*_{ij}\dot{\epsilon}^*_{ij} \tag{4.9}$$

whatever $\dot{\epsilon}^*_{ij}$ so that

$$\sigma^*_{ij} = \frac{\rho^*_0}{\rho_0}\frac{\partial \epsilon_{kl}}{\partial \epsilon^*_{ij}}\sigma_{kl}. \tag{4.10}$$

Further relationships are provided by the one-one correspondence between ϵ_{ij} and ϵ^*_{ij} combined with the elasticity relations (4.4) and (4.6) in both systems. In particular,

$$\epsilon_{ij} = \epsilon_{ij}(\sigma^*_{kl}, p) \tag{4.11}$$

$$\sigma^*_{ij} = \sigma^*_{ij}(\sigma_{kl}, p). \tag{4.12}$$

The infinitesimal cycle of σ^*_{ij} at yield that produces $d\epsilon^{*p}_{ij}$ leaves an increment of ϵ_{ij} equal to $(\partial\epsilon_{ij}/\partial p)(\sigma^*_{kl}, p)\,dp$ and different from $d\epsilon^p_{ij}$ in general. In terms of rates and using (4.8) and (4.12),

$$\dot{\epsilon}^p_{ij} = \left(\frac{\partial \epsilon_{ij}}{\partial p}\right)_\sigma \dot{p} = \left(\frac{\partial \epsilon_{ij}}{\partial p}\right)_{\sigma^*} \dot{p} + \frac{\partial \epsilon_{ij}}{\partial \sigma^*_{mn}}\left(\frac{\partial \sigma^*_{mn}}{\partial p}\right)_\sigma \dot{p} \tag{4.13}$$

where the notation $(\partial\epsilon_{ij}/\partial p)_\sigma$ indicates $(\partial\epsilon_{ij}/\partial p)(\sigma_{kl}, p)$. Evaluating $(\partial\sigma^*_{mn}/\partial p)_\sigma$ from (4.10),

$$\left(\frac{\partial \epsilon_{ij}}{\partial p}\right)_{\sigma^*} \dot{p} = \left[\delta_{ik}\delta_{jl} - \frac{\partial \epsilon_{ij}}{\partial \sigma^*_{mn}}\frac{\partial}{\partial \epsilon_{kl}}\left(\frac{\rho^*_0}{\rho_0}\frac{\partial \epsilon_{rs}}{\partial \epsilon^*_{mn}}\right)\sigma_{rs}\right]\dot{\epsilon}^p_{kl}. \tag{4.14}$$

The second term in the bracket of (4.14) represents the fractional difference between $(\partial\epsilon_{ij}/\partial p)_{\sigma^*}\dot{p}$ and $\dot{\epsilon}^p_{ij}$. With $(\partial/\partial\epsilon_{kl})((\rho^*_0/\rho_0)(\partial\epsilon_{rs}/\partial\epsilon^*_{mn})$ of the order of unity, the effect is of the order of the current stress level divided by the elastic modulus.

Another form of the stability requirement (4.2) is needed for the sequel. Consider the stress cycle *obco*, Fig. 2. Changes in p occur along bc only, from p^b to p^c. Integrating over the cycle and using (4.6),

$$\int(\sigma_{ij} - \sigma^o_{ij})\,d\epsilon_{ij} = -\int \epsilon_{ij}\,d\sigma_{ij} = \int\left(-d\psi + \frac{\partial\psi}{\partial p}dp\right)$$
$$= -\psi(\sigma^o_{ij}, p^c) + \psi(\sigma^o_{ij}, p^b) + \int_b^c \frac{\partial\psi}{\partial p}dp. \tag{4.15}$$

Dividing both sides by $p^c - p^b$ and letting c tend to b as it must for infinitesimal plastic straining, (4.2) becomes

$$\frac{\partial\psi}{\partial p}(\sigma^b_{ij}, p) - \frac{\partial\psi}{\partial p}(\sigma^o_{ij}, p) \geq 0 \tag{4.16}$$

where p stands for p^b. Invariance of (4.2) upon a change of stress and strain measures and of reference state is necessary for material stability as stated here to make sense. It can be proved from (4.16). Total differentiation of (4.5) yields

$$\frac{\partial \psi}{\partial p} = -\frac{\partial \phi}{\partial p}. \tag{4.17}$$

Invariance of $(1/\rho_0)(\partial \phi/\partial p)$ carries over to $(1/\rho_0)(\partial \psi/\partial p)$, to (4.16) and hence (4.2). The equivalence of (3.1) and (4.1) follows by analogy[18]. In the rigid plastic case, invariance of (4.3) is proved still more easily by use of eqn (4.10) where the coefficients relating the two stress definitions are constant inside the yield surface.

5. NORMALITY. THE CASE OF NON-INTEGRABLE STRAIN RATES

Normality of $\dot{\epsilon}_{ij}^{p}$ in systems of symmetric, conjugate measures of stress and strain was shown in [2] to follow from Ilyushin's postulate and so also follows for the class of materials considered here. Let o tend to b along an arbitrary path inside the yield surface. Equation (4.16) becomes, in the limit,

$$\delta\sigma_{ij} \frac{\partial}{\partial \sigma_{ij}} \left(\frac{\partial \psi}{\partial p}\right)(\sigma_{kl}^{b}, p) \geq 0 \tag{5.1}$$

where $\delta\sigma_{ij}$ is an arbitrary stress increment from the interior of the yield surface to the current yield point. Multiplying by $\dot{p}$ and using (4.8),

$$\delta\sigma_{ij}\dot{\epsilon}_{ij}^{p} \geq 0. \tag{5.2}$$

Normality in stress space follows at a regular point on the yield surface and extended normality at a corner[13], Fig. 3. An inward moving yield surface causes no difficulty in the present proof of normality based on material stability. Limits are sequential, first c, then o tend to b, Fig. 2. However close to b o may be, a stress cycle can always be closed for a small enough excursion into the plastic range.

Cauchy and Kirchhoff stresses computed in rectangular Cartesian axes rotating in the continuum sense with the body element[3, 19] are examples of symmetric and objective definitions of stress with conjugate "strain rates" that cannot be integrated into a strain measure. Using an arbitrary reference state, those corotational Cauchy and Kirchhoff stresses are given, respectively, by

$$R_{ki}R_{lj}T_{kl} \text{ and } \frac{\rho_0}{\rho} R_{ki}R_{lj}T_{kl}$$

where T_{kl} are the components of Cauchy stress in the fixed axes, (ρ_0/ρ) is the ratio of the densities in the reference and current states, and R_{ij} is the rotation tensor defined by the polar decomposition of the deformation gradient[17]. The conjugate, non-integrable strain rates are the components of the rate-of-deformation tensor on the rotated axes for the corotational Kirchhoff stress and the same, times (ρ_0/ρ) for the corotational Cauchy stress.

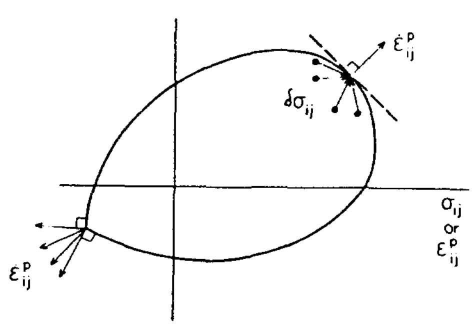

Fig. 3. Normality of plastic strain rate in stress space.

Let σ^*_{ij} be an arbitrary symmetric and objective stress with conjugate non-integrable strain rate $\dot{\epsilon}^*_{ij}$. Investigation of normality in σ^*_{ij} space requires the introduction of an auxiliary system of symmetric stress σ_{ij} and conjugate strain ϵ_{ij} to define the elastic response. The reference state is taken to be the same in both systems. Because of objectivity, σ_{ij} and σ^*_{ij} are related through components of strain only,

$$\sigma_{ij} = A_{ijkl}(\epsilon_{mn})\sigma^*_{kl}. \tag{5.3}$$

The rate of work equality (4.9) with $\rho_0 = \rho^*_0$ gives

$$\dot{\epsilon}^*_{ij} = A_{klij}\dot{\epsilon}_{kl}. \tag{5.4}$$

Non-integrability means

$$\frac{\partial A_{klij}}{\partial \epsilon_{mn}} \neq \frac{\partial A_{mnij}}{\partial \epsilon_{kl}}. \tag{5.5}$$

Let $(\partial\psi/\partial p)$ be evaluated in the σ_{ij} system and then written in terms of σ^*_{ij} and p by the inverse of (4.12). By the same argument as at the beginning of this section,

$$\dot{\epsilon}^{*n}_{ij} = \frac{\partial}{\partial \sigma^*_{ij}}\left(\frac{\partial\psi}{\partial p}\right)\dot{p} \tag{5.6}$$

is normal to the yield surface in σ^*_{ij} space. On the other hand, substitution of (4.11) into (5.4) gives

$$\dot{\epsilon}^*_{ij} = A_{klij}\frac{\partial \epsilon_{kl}}{\partial \sigma^*_{mn}}\dot{\sigma}^*_{mn} + A_{klij}\left(\frac{\partial \epsilon_{kl}}{\partial p}\right)_{\sigma^*}\dot{p}. \tag{5.7}$$

The first term on the r.h.s. of (5.7) is the strain rate produced by $\dot{\sigma}^*_{mn}$ in a purely elastic response. According to the definition adopted here, the second term is the plastic strain rate

$$\dot{\epsilon}^{*p}_{ij} = A_{klij}\left(\frac{\partial \epsilon_{kl}}{\partial p}\right)_{\sigma^*}\dot{p}. \tag{5.8}$$

The relationship between $\dot{\epsilon}^{*n}_{ij}$ and $\dot{\epsilon}^{*p}_{ij}$ is derived as follows. From (5.6) and (4.8),

$$\dot{\epsilon}^{*n}_{ij} = \frac{\partial \sigma_{rs}}{\partial \sigma^*_{ij}}\dot{\epsilon}^p_{rs}. \tag{5.9}$$

By steps similar to those leading to eqn (4.14),

$$\dot{\epsilon}^p_{rs} = \left(\delta_{rk}\delta_{sl} - \frac{\partial \epsilon_{rs}}{\partial \sigma_{mn}}\frac{\partial A_{mnpq}}{\partial \epsilon_{kl}}\sigma^*_{pq}\right)\left(\frac{\partial \epsilon_{kl}}{\partial p}\right)_{\sigma^*}\dot{p}. \tag{5.10}$$

Substitution of (5.10) in (5.9), use of (5.3), (5.8) and

$$\frac{\partial \epsilon_{rs}}{\partial \sigma_{mn}} = \frac{\partial \epsilon_{mn}}{\partial \sigma_{rs}} \tag{5.11}$$

gives

$$\dot{\epsilon}^{*n}_{ij} - \dot{\epsilon}^{*p}_{ij} = \left(\frac{\partial A_{klpq}}{\partial \epsilon_{mn}} - \frac{\partial A_{mnpq}}{\partial \epsilon_{kl}}\right)\frac{\partial \epsilon_{mn}}{\partial \sigma^*_{ij}}\sigma^*_{pq}A^{-1}_{rskl}\dot{\epsilon}^{*p}_{rs}. \tag{5.12}$$

Non-integrability, eqn (5.5), enforces $\dot{\epsilon}^{*n}_{ij} \neq \dot{\epsilon}^{*p}_{ij} \cdot \dot{\epsilon}^{*p}_{ij}$ is not normal to the yield surface in σ^*_{ij}

space in general. Lack of normality is quantitatively measured as follows. Let the norm of a tensor a_{ij} be the usual

$$\|a_{ij}\| = (a_{ij}a_{ji})^{1/2}. \tag{5.13}$$

Consider a symmetric unit tensor m_{ij} tangent to the yield surface and terminating at the point under consideration, Fig. 4. If a_{ij} is normal to the yield surface, $m_{ij}a_{ij}$ is zero at a regular point and positive or zero at a singular point. Normality at a regular or a singular point is not satisfied if $m_{ij}a_{ij}$ is negative for some m_{ij}. In that case, the extent of the lack of normality is defined as

$$\sup\left\{-m_{ij}\frac{a_{ij}}{\|a_{kl}\|}\right\} \tag{5.14}$$

over all possible m_{ij}, where sup stands for least upper bound. At a regular point, (5.14) gives $\sin\alpha$, where α is the angle between a_{ij} and the normal to the yield surface, Fig. 4. Given normality of $\dot{\epsilon}_{ij}^{*n}$,

$$-m_{ij}\dot{\epsilon}_{ij}^{*p} \le m_{ij}(\dot{\epsilon}_{ij}^{*n} - \dot{\epsilon}_{ij}^{*p}). \tag{5.15}$$

It follows from (5.12) and (5.15) that the lack of normality of $\dot{\epsilon}_{ij}^{*p}$ is of the order of the current stress level divided by Young's modulus. Still smaller deviations occur if σ_{ij}^* is the corotational Kirchhoff stress. This stress definition differs from the stress conjugate to logarithmic strain by second order terms in strain[3]. For that pair of stress measures $(\partial A_{klpq}/\partial\epsilon_{mn}) - (\partial A_{mnpq}/\partial\epsilon_{kl})$ in eqn (5.12) is of first order in strain. Lack of normality is of the order of the square of the ratio of current stress to elastic modulus if the reference is in the unloaded state of the body element. Normality is satisfied exactly if the reference is taken in the current deformed state.

6. CONVEXITY

Existing information on convexity of the yield surface at finite strain is restricted to special cases[5, 6]. Equation (4.16) here provides the basis for a general investigation. By application of Taylor's formula around σ_{ij}^b to (4.16),

$$\frac{\partial}{\partial\sigma_{ij}}\left(\frac{\partial\psi}{\partial p}\right)^b(\sigma_{ij}^b - \sigma_{ij}^o) \ge \frac{1}{2}\frac{\partial^2}{\partial\sigma_{ij}\partial\sigma_{kl}}\left(\frac{\partial\psi}{\partial p}\right)^q(\sigma_{ij}^b - \sigma_{ij}^o)(\sigma_{kl}^b - \sigma_{kl}^o) \tag{6.1}$$

where q lies on the segment joining o and b in stress space and indices b and q in the derivatives of ψ indicate arguments σ_{ij}^b and σ_{ij}^q. Validity of (6.1) may require extending the elastic response outside the yield surface, at least to its convex hull, Fig. 5. Using (4.8) and reversing the order of differentiation on the r.h.s. of (6.1),

$$(\sigma_{ij}^b - \sigma_{ij}^o)\dot{\epsilon}_{ij}^p \ge \frac{1}{2}\frac{\partial}{\partial p}\left(\frac{\partial^2\psi}{\partial\sigma_{ij}\partial\sigma_{kl}}\right)^q\dot{p}(\sigma_{ij}^b - \sigma_{ij}^o)(\sigma_{kl}^b - \sigma_{kl}^o) \tag{6.2}$$

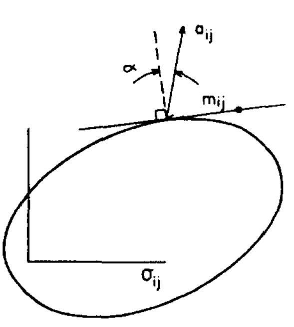

Fig. 4. Lack of normality.

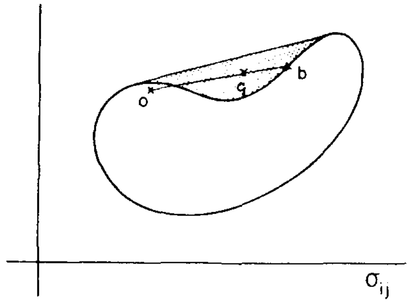

Fig. 5. An extended elastic response may be required outside the yield surface.

where $(\partial^2\psi/\partial\sigma_{ij}\partial\sigma_{kl})$ are the elastic compliances. Given normality of $\dot{\epsilon}^p_{ij}$ and restricting b to be a regular point, the yield surface in stress space is convex if $(\sigma^b_{ij} - \sigma^o_{ij})\dot{\epsilon}^p_{ij}$ is non-negative for all pairs b, o. From (6.2) a sufficient criterion for convexity is that $(\partial/\partial p)(\partial^2\psi/\partial\sigma_{ij}\partial\sigma_{kl})\dot{p}$, the rate of change of the tensor of elastic compliances at a fixed value of stress and at a fixed reference state, be positive semi-definite in some convex region containing the elastic domain. Zero change of the elastic compliances trivially meets the requirement. This was the case in [6] where a linear elastic response in Piola–Kirchhoff II stress space was unaltered by plastic deformation. Yet zero change of the elastic response at a fixed reference state is a property which does not carry over in finite elasto-plasticity from one stress definition to another. The "true" or physical change of elastic response is computed from the unloaded geometry of the material element[9], a changing reference state. The Appendix relates the two types of changes. $(\partial/\partial p)(\partial^2\psi/\partial\sigma_{ij}\partial\sigma_{kl})\dot{p}$ is shown to be of the order of the elastic compliance times the plastic strain rate for a negligible "true" modulus effect such as that of usual metals and alloys. It is established that $(\partial/\partial p)(\partial^2\psi/\partial\sigma_{ij}\partial\sigma_{kl})\dot{p}$ transforms linearly from one stress system to another when the "true" modulus effect is large. The present criterion for convexity is then expected to be most useful.

Convexity needs not hold in general. Deviations from convexity are possible, which are characterized by

$$n_{ij}(\sigma^b_{ij} - \sigma^o_{ij}) < 0 \tag{6.3}$$

for some regular point b on the yield surface with outer unit normal n_{ij} and some interior point o. In that case, the extent of the lack of convexity is defined as

$$\sup\{-n_{ij}(\sigma^b_{ij} - \sigma^o_{ij})\} \tag{6.4}$$

over all possible pairs b, o. For a given b, (6.4) is the distance d indicated in Fig. 6. The lack of convexity is the least upper bound of such distances as b moves along the surface. From (6.2) deviations from convexity are bounded,

$$-n_{ij}(\sigma^b_{ij} - \sigma^o_{ij}) \leq -\frac{1}{2}\frac{1}{\|\dot{\epsilon}^p_{mn}\|}\frac{\partial}{\partial p}\left(\frac{\partial^2\psi}{\partial\sigma_{ij}\partial\sigma_{kl}}\right)^q \dot{p}(\sigma^b_{ij} - \sigma^o_{ij})(\sigma^b_{kl} - \sigma^o_{kl}). \tag{6.5}$$

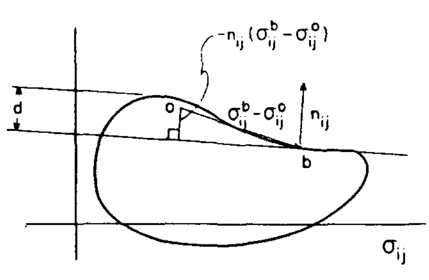

Fig. 6. Lack of convexity.

The possible lack of convexity for usual metals and alloys is of the order of the square of the diameter of the yield surface divided by the elastic modulus. It is not likely to be observed experimentally and is neglected in the usual infinitesimal treatment of elasticity.

Convexity does not fully transfer from one system of stress σ_{ij} to another σ^*_{ij} or from the load to the arbitrary stress space of the tube specimen. Let $f = 0$ be the equation of the yield surface. From the inverse of (4.12) and evaluation of the gradient of f at the regular point b under consideration,

$$\frac{\partial f}{\partial \sigma_{ij}}(\sigma^b_{ij} - \sigma^o_{ij}) - \frac{\partial f}{\partial \sigma^*_{kl}}(\sigma^{*b}_{kl} - \sigma^{*o}_{kl}) = -\frac{\partial f}{\partial \sigma_{ij}}\left[\sigma^o_{ij} - \sigma^b_{ij} - \left(\frac{\partial \sigma_{ij}}{\partial \sigma^*_{kl}}\right)^b (\sigma^{*o}_{kl} - \sigma^{*b}_{kl})\right]$$

$$= -\frac{1}{2}\frac{\partial f}{\partial \sigma_{ij}}\left(\frac{\partial^2 \sigma_{ij}}{\partial \sigma^*_{kl}\partial \sigma^*_{mn}}\right)^s (\sigma^{*b}_{kl} - \sigma^{*o}_{kl})(\sigma^{*b}_{mn} - \sigma^{*o}_{mn}) \tag{6.6}$$

where sufficient smoothness has been assumed for Taylor's formula. s is some intermediate point on the segment ob in stress space. $(\partial^2\sigma_{ij}/\partial\sigma^*_{kl}\partial\sigma^*_{mn})$ is of the order of the reciprocal of the elastic modulus as shown in the Appendix when $\sigma_{ij}, \sigma^*_{ij}$ have conjugate strain measures and similarly if one has a conjugate non-integrable strain rate. If the yield surface is convex in σ_{ij} space it follows from (6.6) that the maximum possible lack of convexity in σ^*_{ij} space is of the order of the square of the diameter of the yield surface divided by the elastic modulus.

In concluding the sections on normality and convexity it is again worth emphasizing the simplicity brought about by the plastic-rigid idealization. Normality holds strictly in systems with non-integrable strain rates. Convexity carries over exactly from one stress system to another.

Stability considerations have been applied to symmetric stresses with a conjugate strain measure. Non-symmetric stresses of the Piola type[17], with conjugate quantities comprising both strain and rotation could also be used in principle. The only troublesome point is that the invertibility of the elastic stress–strain relations (4.6) cannot be taken for granted[20]. The problem is minor for normality as local invertibility is sufficient. However, global invertibility is required for convexity.

7. THE DEFINITION OF PLASTIC STRAIN OR PLASTIC STRAIN RATE

A popular approach today is to define as "plastic" the permanent or residual strain remaining after elastically unloading the body element to zero stress. Unloading is performed without rotation for isotropic material[9] or with rotation for plastic anisotropy and isoclinic stress-free configurations[11]. The change in the stress-free state is viewed as directly corresponding to the migration of dislocations and other defects responsible for plastic deformation on the microscale. One difficulty is that the rate of the permanent or residual strain is not normal to the yield surface in general.

It is the total strain rate minus the elastic strain rate that is normal to the yield surface in stress space and defined here as plastic strain rate. The elastic strain rate must be computed from the stress rate and the instantaneous elastic compliances at the current state of stress. This was pointed out by Palmer, Maier and Drucker in the infinitesimal strain case when elastic properties are altered by plastic deformation[16]. Cauchy stress referred to fixed axes cannot be used to define a plastic strain rate because it is symmetric and non-objective. As discussed in [10, 21] an infinitesimal cycle of this type of stress leaves an Eulerian increment of strain that incorporates erroneous rotation effects. The difference between the two approaches to the definition of plastic strain or strain rate is illustrated in Fig. 7 for an arbitrary uniaxial stress σ and conjugate strain ϵ.

The question of whether the increment of plastic strain $d\epsilon^p_{ij}$ represents "true" plastic deformation or contains elastic contributions is irrelevant for normality. In fact what is taken here to be plastic depends upon the chosen stress measure as shown in the section on material stability. The analogous still more striking case of a structure acted upon by forces P_i clarifies this mathematical and physical point further[14]. Application and removal of dP_i at yield leaves conjugate displacements du_i^p which are normal to the yield surface in load space. Yet most of the structure remains elastic while part deforms plastically during the application of dP_i so that

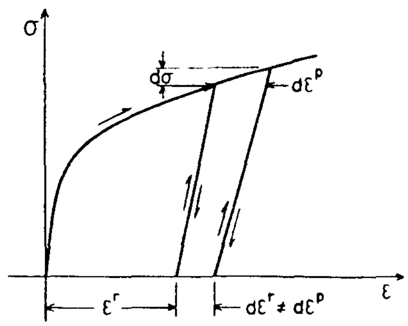

Fig. 7. Increment of plastic strain $d\epsilon^p$ and permanent or residual strain ϵ^r.

du_i^p contains elastic residuals. *A fortiori* the sum of the successive $d\epsilon_{ij}^p$ or du_i^p need not be physically interpretable as "plastic".

Similarly an unloaded element of material is unlikely to have sustained pure plastic deformation in the physical sense. Large elastic distortions are induced locally on the micro-scale as dislocations pile-up against inclusions or grain boundaries, or get caught in tangles. These and other effects involve changes in local residual microstresses far in excess of the macroscopic yield stress of the material.

Explicit evaluation of the difference between the present plastic strain rate $\dot{\epsilon}_{ij}^p$ and the rate of permanent or residual strain $\dot{\epsilon}_{ij}^r$ follows. The present treatment generalizes that given in [22] for a linear elastic response. The permanent or residual strain is given by (4.6) with σ_{kl} set equal to zero. By additional use of (4.8) and Taylor's formula,

$$\dot{\epsilon}_{ij}^r - \dot{\epsilon}_{ij}^p = \frac{\partial \epsilon_{ij}}{\partial p}(0, p)\dot{p} - \frac{\partial \epsilon_{ij}}{\partial p}(\sigma_{mn}, p)\dot{p}$$
$$= -\frac{\partial}{\partial p}\left(\frac{\partial \epsilon_{ij}}{\partial \sigma_{kl}}\right)(\theta\sigma_{mn}, p)\dot{p}\sigma_{kl} \quad 0 < \theta < 1 \tag{7.1}$$

where partial derivatives with respect to p and σ_{kl} have been interchanged. As in convexity considerations $(\partial/\partial p)(\partial\epsilon_{ij}/\partial\sigma_{kl})\dot{p}$ is the rate of change of the elastic compliances at a fixed value of stress and at a fixed reference state. From (7.1) and the normality of $\dot{\epsilon}_{ij}^p$, the lack of normality of $\dot{\epsilon}_{ij}^r$ according to the definition (5.14) is of the order of the change of elastic compliances with permanent or residual strain, $(1/\|\dot{\epsilon}_{mn}^r\|)(\partial/\partial p)(\partial\epsilon_{ij}/\partial\sigma_{kl})\dot{p}$, times the current stress level. The effect can be of importance in some composites or in materials whose microstructure is severely disturbed by plastic deformation.

An entirely different approach to the definition of plastic strain or strain rate is to choose plastic strain as an internal variable[19]. Normality no longer holds in general. In the cases where normality was reported[6, 23], the rate of the internal variable chosen to be plastic strain did coincide with the plastic strain rate as defined here.

8. CONCLUSIONS

Torsion of thin-walled tubes demonstrates the need in stress–strain relations to distinguish rotation in the material and continuum sense.

Material stability is a direct generalization of stability in Drucker's sense of statically determinate test specimens under load such as thin-walled tubes under combined tension, torsion and interior pressure. Conjugate measures of stress and strain provide the appropriate framework. Within the constraint of stability, the yield surface in stress space must be convex if the rate of change of the tensor of elastic compliances at a fixed value of stress and at a fixed reference state is positive semi-definite.

In the presence of a large modulus effect, normality does not hold for the rate of the permanent or residual strain in the stress-free state. It is the total strain rate minus the elastic strain rate computed from the stress rate and the instantaneous elastic compliances at the current stress that is normal to the yield surface. Lack of convexity is possible if the elastic response stiffens considerably with plastic deformation.

Many of the effects discussed here are of the order of the current stress level or diameter of the yield surface divided by the elastic modulus. These include the lack of normality with non-integrable strain rate, the effect on convexity of changing the stress definition or, for a small change of elastic response, the difference between permanent and plastic strain rates and the possible lack of convexity of the yield surface compatible with stability. All these effects would not be observable experimentally with any certainty for usual metals and alloys whose elastic or small offset yield range is well below 0.01. They can be greater for very high strength steels, special alloys and polymeric structural materials which have two or three times the elastic range of usual metals and alloys and some day could conceivably come close to the theoretical strength limits of about 0.1 elastic strain. Useful alloys are now at strength levels at which the hydrostatic tension or compression measurably affects the yield and flow strengths. Destabilizing effects similar to those in frictional systems that may result have been ignored here.

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APPENDIX

The change of elastic compliances at a fixed reference state

The purpose of this Appendix is to estimate the change of elastic compliances at a fixed reference state from the "true" or physical change of elastic response with respect to the changing unloaded state of the material element. The unloaded state is chosen to correspond to a zero lattice or material rotation. The stress definitions here are symmetric with a conjugate strain measure. First computations are made for the Piola-Kirchhoff II stress tensor S and conjugate Green strain E[17], taking the fixed reference in the instantaneous unloaded state. The direct notation for tensors[17] is used for convenience. Let $\mathbf{F}$ denote the deformation gradient from the fixed reference state. The multiplicative decomposition of $\mathbf{F}$ is[9]

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{A1}$$

where $\mathbf{F}^p$ is instantaneously equal to the identity tensor $\mathbf{I}$. Let $\mathbf{C}$ be the right Cauchy strain tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = 2\mathbf{E} + \mathbf{I} \tag{A2}$$

and $\mathbf{C}^e$ be similarly defined from $\mathbf{F}^e$. Equality of the strain energies per unit mass computed with respect to the fixed and changing reference states gives

$$\frac{1}{\rho_0}\phi(\mathbf{C}, p) = \frac{1}{\rho_0'}\phi'(\mathbf{C}^e, p) \tag{A3}$$

where a prime applies to the changing geometry. ρ_0, ρ_0' are the respective mass densities in the reference states. From (A1)–(A3) and

$$\mathbf{S} = 2\frac{\partial\phi}{\partial\mathbf{C}^e} \tag{A4}$$

instantaneously, it can be shown[18] that, at the current time,

$$\frac{\partial\phi}{\partial p}\dot{p} = -\mathrm{tr}(\mathbf{S}\mathbf{C}\dot{\mathbf{F}}^p) + \phi\,\mathrm{tr}\,\dot{\mathbf{F}}^p + \frac{\partial\phi'}{\partial p}\dot{p}. \tag{A5}$$

Use of (4.17) and double differentiation with respect to $\mathbf{S}$ assuming sufficient smoothness gives

$$\frac{\partial}{\partial p}\left(\frac{\partial^2\psi}{\partial\mathbf{S}\partial\mathbf{S}}\right)\dot{p} = \frac{\partial^2}{\partial\mathbf{S}\partial\mathbf{S}}[\mathrm{tr}(\mathbf{S}\mathbf{C}\dot{\mathbf{F}}^p)] - \frac{\partial}{\partial\mathbf{S}}\left(\frac{\partial\mathbf{E}}{\partial\mathbf{S}}\mathbf{S}\right)\mathrm{tr}\,\dot{\mathbf{F}}^p + \frac{\partial}{\partial p}\left(\frac{\partial^2\psi'}{\partial\mathbf{S}\partial\mathbf{S}}\right)\dot{p} \tag{A6}$$

where partial derivatives with respect to p and $\mathbf{S}$ have been interchanged. $(\partial/\partial p)(\partial^2\psi/\partial\mathbf{S}\partial\mathbf{S})\dot{p}$ is the rate of change of elastic compliances at a fixed value of stress and at a fixed reference state. It is composed of the "true" rate of change of elastic compliances, $(\partial/\partial p)(\partial^2\psi'/\partial\mathbf{S}\partial\mathbf{S})\dot{p}$ and of the effect of using a fixed reference state, represented by the other terms on the r.h.s. of (A6). The latter contribution is of the order of the elastic compliance times the permanent or residual strain rate for a usual elastic response close to linear. Close to linear here means that the stress derivative of the elastic compliance times the current stress level is of the order of the elastic compliance or smaller. When the contribution of the "true" modulus effect is not large, the plastic strain rate as defined here and the permanent or residual strain rate do not differ appreciably, eqn (7.1), and $(\partial/\partial p)(\partial^2\psi/\partial\mathbf{S}\partial\mathbf{S})\dot{p}$ is of the order of the elastic compliance times the plastic strain rate.

The above conclusions now will be extended to any stress definition with an arbitrary reference state. This amounts to examining how $(\partial/\partial p)(\partial^2\psi/\partial\sigma_{ij}\partial\sigma_{kl})\dot{p}$ transforms from one stress definition σ_{ij} to another σ^*_{ij}. By invariance of $(1/\rho_0)(\partial\psi/\partial p)$.

$$\frac{1}{\rho_0^*}\frac{\partial\psi^*}{\partial p}\dot{p} = \frac{1}{\rho_0}\frac{\partial\psi}{\partial p}\dot{p}. \tag{A7}$$

Double differentiation with respect to $\sigma^*_{ij}, \sigma^*_{kl}$ assuming sufficient smoothness gives, by use of (4.8), (4.12) and its inverse,

$$\frac{\partial}{\partial p}\left(\frac{\partial^2\psi^*}{\partial\sigma^*_{ij}\partial\sigma^*_{kl}}\right)\dot{p} = \frac{\rho_0^*}{\rho_0}\frac{\partial}{\partial p}\left(\frac{\partial^2\psi}{\partial\sigma_{mn}\partial\sigma_{rs}}\right)\dot{p}\,\frac{\partial\sigma_{mn}}{\partial\sigma^*_{ij}}\frac{\partial\sigma_{rs}}{\partial\sigma^*_{kl}} + \frac{\partial^2\sigma_{mn}}{\partial\sigma^*_{ij}\partial\sigma^*_{kl}}\frac{\partial\sigma^*_{rs}}{\partial\sigma_{mn}}\dot{\epsilon}^{*p}_{rs} \tag{A8}$$

where partial derivatives with respect to p and stress components have been interchanged. From the inverse of (4.10), $(\partial\sigma_{mn}/\partial\sigma^*_{ij})$ is of the order of unity and

$$\begin{aligned}\frac{\rho_0^*}{\rho_0}\frac{\partial^2\sigma_{mn}}{\partial\sigma^*_{ij}\partial\sigma^*_{kl}} = {}& \frac{\partial}{\partial\epsilon^*_{rs}}\left(\frac{\partial\epsilon^*_{ij}}{\partial\epsilon_{mn}}\right)\frac{\partial\epsilon^*_{rs}}{\partial\sigma^*_{kl}} + \frac{\partial}{\partial\epsilon^*_{rs}}\left(\frac{\partial\epsilon^*_{kl}}{\partial\epsilon_{mn}}\right)\frac{\partial\epsilon^*_{rs}}{\partial\sigma^*_{ij}} + \frac{\partial}{\partial\epsilon^*_{rs}}\left(\frac{\partial\epsilon^*_{pq}}{\partial\epsilon_{mn}}\right)\frac{\partial^2\epsilon^*_{rs}}{\partial\sigma^*_{ij}\partial\sigma^*_{kl}}\sigma^*_{pq} \\ & + \frac{\partial^2}{\partial\epsilon^*_{rs}\partial\epsilon^*_{uv}}\left(\frac{\partial\epsilon^*_{pq}}{\partial\epsilon_{mn}}\right)\frac{\partial\epsilon^*_{rs}}{\partial\sigma^*_{ij}}\frac{\partial\epsilon^*_{uv}}{\partial\sigma^*_{kl}}\sigma^*_{pq}\end{aligned} \tag{A9}$$

is of the order of the elastic compliance for an elastic response close to linear. If the "true" modulus effect is not dominant in (A6), it follows from the discussion of that equation and from (A8) that $(\partial/\partial p)(\partial^2\psi/\partial\sigma_{ij}\partial\sigma_{kl})\dot{p}$ is of the order of the elastic compliance times the plastic strain rate whatever the stress definition and reference state. For a dominant "true" modulus effect the second term on the r.h.s. of (A8) can be neglected. The rate of change of elastic compliances transforms linearly from one stress system to another or, analogously, from the load to the arbitrary stress space of the tube specimen.